Quantum Mechanics

Angular momentum operator in Cartesian Coordinate

In classical mechanics, the orbital angular momentum of a particle relative to the Centre of rotation is a vector quantity L. The orbital angular momentum, L is given by equation

$$L = r \times p \quad \text{——— (1)}$$

Vector L is perpendicular to the plane formed by the vector r and p. where p is the linear momentum of the particle and r is the position vector of the particle from the axis of rotation. The Component of angular momentum L about x, y, and z axis is

$$\vec{L} = iL_x + jL_y + kL_z \text{, Also for}$$

$$\vec{r} = ix + jy + kz \quad \text{——— (2)}$$

$$\vec{p} = ip_x + jp_y + kp_z$$

Putting in (1), we have

$$iL_x + jL_y + kL_z = (ix + jy + kz) \times (ip_x + jp_y + kp_z)$$

$$= i(yp_z - zp_y) + j(zp_x - xp_z) + k(xp_y - yp_x)$$

Thus, the Component of angular momenta L_x, L_y and L_z along x, y, and z axis are

$$L_x = yp_z - zp_y$$

$$L_y = zp_x - xp_z \qquad \text{or } L = \begin{vmatrix} i & j & k \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} \quad \text{— (3)}$$

$$L_z = xp_y - yp_x$$

Since $p = \frac{\hbar}{i}\frac{\partial}{\partial r}$, we have

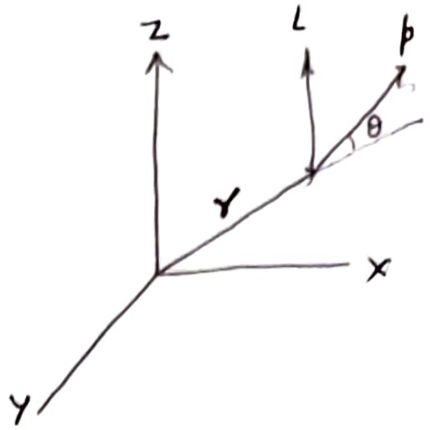
$$p_x = \frac{\hbar}{i}\frac{\partial}{\partial x} \; ; \; p_y = \frac{\hbar}{i}\frac{\partial}{\partial y} \; ; \; p_z = \frac{\hbar}{i}\frac{\partial}{\partial z} \quad \text{— (4)}$$

we have,

$$L_x = y\left(\frac{\hbar}{i}\frac{\partial}{\partial z}\right) - z\left(\frac{\hbar}{i}\frac{\partial}{\partial y}\right)$$

$$L_y = z\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right) - x\left(\frac{\hbar}{i}\frac{\partial}{\partial z}\right)$$

$$L_z = x\left(\frac{\hbar}{i}\frac{\partial}{\partial y}\right) - y\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)$$



$$\hat{L}_x = -i\hbar\left[y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right]$$

$$\hat{L}_y = -i\hbar\left[z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right]$$

$$\hat{L}_z = -i\hbar\left[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right]$$

<u>Angular momentum operator in Spherical Coordinates</u>

<u>$r, \theta, \& \phi$</u>

$$x = r\sin\theta\cos\phi \qquad \text{——— (1)}$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

Again. $r^2 = x^2+y^2+z^2$; $\cos\theta = \frac{z}{r}$ and $\tan\phi = \frac{y}{x}$

Now.

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial}{\partial\theta}\cdot\frac{\partial\theta}{\partial x} + \frac{\partial}{\partial\phi}\cdot\frac{\partial\phi}{\partial x} \qquad \text{——— (2a)}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial}{\partial\theta}\cdot\frac{\partial\theta}{\partial y} + \frac{\partial}{\partial\phi}\cdot\frac{\partial\phi}{\partial y} \qquad \text{——— (2b)}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial}{\partial\theta}\cdot\frac{\partial\theta}{\partial z} + \frac{\partial}{\partial\phi}\cdot\frac{\partial\phi}{\partial z} \qquad \text{——— (2c)}$$

Equation (2a) may be put as follows:-

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial}{\partial\theta}\cdot\frac{\partial\theta}{\partial x} + \frac{\partial}{\partial\phi}\cdot\frac{\partial\phi}{\partial x}$$

When $r^2 = x^2+y^2+z^2$ is differentiated with respect to x, we get

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \frac{\cancel{r}\sin\theta\cos\phi}{\cancel{r}} = \sin\theta\cos\phi \quad -(3)$$

$$\cos\theta = \frac{z}{r} = \frac{z}{(x^2+y^2+z^2)^{1/2}}$$

Differentiating the above eqn. with respect to x, we get

$$-\sin\theta\frac{\partial\theta}{\partial x} = -\frac{1}{\cancel{2}}\frac{z.\cancel{2}x}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial\theta}{\partial x} = \frac{r\cos\theta . r\sin\theta\cos\phi}{r^3\sin\theta}$$

$$\frac{\partial\theta}{\partial x} = \frac{\cos\theta\cos\phi}{r} \quad \text{———} \quad (4)$$

But $\tan\phi = \frac{y}{x}$

$$\therefore \quad \sec^2\phi\frac{\partial\phi}{\partial x} = \frac{-y}{x^2} = \frac{-r\sin\theta\sin\phi}{r^2\sin^2\theta\cos^2\phi}$$

$$\frac{\partial\phi}{\partial x} = \frac{-r\sin\theta\sin\phi\,\cancel{\cos^2\phi}}{r^2\sin^2\theta\,\cancel{\cos^2\phi}} = -\frac{1}{r}\frac{\sin\phi}{\sin\theta} \quad -(5)$$

Putting the value of $\frac{\partial r}{\partial x}$, $\frac{\partial\phi}{\partial x}$ & $\frac{\partial\theta}{\partial x}$ in eqn. (2a), we get

$$\frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi} \quad \text{———} \quad (6)$$

Similarly, it can be proved that

$$\frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi} \quad \text{———} \quad (7)$$

$$\& \quad \frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta} \quad \text{———} \quad (8)$$

Now.

$$\hat{L}_x = -i\hbar\left[y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right]$$

$$= -i\hbar\left[r\sin\theta\sin\phi\left(\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right) - r\cos\theta\left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)\right]$$

$$= -i\hbar\left[r\sin\theta\sin\phi\cos\theta\frac{\partial}{\partial r} - \sin^2\theta\sin\phi\frac{\partial}{\partial\theta} - r\cos\theta\sin\theta\sin\phi\frac{\partial}{\partial r} - \cos^2\theta\sin\phi\frac{\partial}{\partial\theta} - \frac{\cos\theta\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right]$$

$$\therefore \hat{L}_x = -i\hbar\left[-\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right]$$

$$\hat{L}_x = +i\hbar\left[\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right]$$

Similarly, it can be proved that

$$\hat{L}_y = i\hbar\left[-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right]$$

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$$

Now. $L^2 = L_x^2 + L_y^2 + L_z^2$

$$L^2 = -\hbar^2\left[\left\{\sin^2\phi\frac{\partial^2}{\partial\theta^2} + \cot^2\theta\cos^2\phi\frac{\partial^2}{\partial\phi^2} + \sin\phi\frac{\partial}{\partial\theta}\left(\cot\theta\cos\phi\frac{\partial}{\partial\phi}\right) + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\theta}\right)\right\}\right.$$
$$+\left\{\cos^2\phi\frac{\partial^2}{\partial\theta^2} + \cot^2\theta\sin^2\phi\frac{\partial^2}{\partial\phi^2} - \cos\phi\frac{\partial}{\partial\theta}\left(\cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)\right.$$
$$\left.\left. - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial}{\partial\theta}\right)\right\} + \left\{\frac{\partial^2}{\partial\phi^2}\right\}\right]$$

$$= -\hbar^2\Big[\frac{\partial^2}{\partial\theta^2} + (1+\cot^2\theta)\frac{\partial^2}{\partial\phi^2} - \sin\phi\,\mathrm{Cosec}^2\theta\,\cos\phi\frac{\partial}{\partial\phi}$$

$$+ \cot\theta\cos^2\phi\frac{\partial}{\partial\theta} + \cos\phi\,\mathrm{Cosec}^2\theta\,\sin\phi\frac{\partial}{\partial\phi}$$

$$+ \cot\theta\sin^2\phi\frac{\partial}{\partial\theta}\Big]$$

$$= -\hbar^2\Big[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + (1+\cot^2\theta)\frac{\partial^2}{\partial\phi^2}\Big]$$

$$\Rightarrow L^2 = -\hbar^2\Big[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial}{\partial\theta}\Big) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\Big]$$

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